MATH 344–Differential Geometry III: Riemannian geometry

Name: ______________________________

SCIPER No: ______________________________

Instructions:

1. This examination contains 5 questions and 14 pages, including this page.
2. You have **two hours and forty five minutes (2:45)** to complete the examination.
3. Write your answers in the blank space between the questions (including on the back of the corresponding pages). If you need additional space, you can use additional blank papers (we will provide them to you), but make sure to write your name on those.
4. You may use one (1) one-sided A4 page with notes that you have prepared. You may not use any other resources, including lecture notes, books, or other students.

1. Let $(\mathcal{M}, g)$ be a connected Riemannian manifold.

   (i) Define the Riemannian distance $\mathrm{dist}_g(p, q)$ between two points $p, q \in \mathcal{M}$.

   (ii) Define when $(\mathcal{M}, g)$ is *geodesically complete* and when it is *complete* (as a metric space).

   (iii) State the Hopf–Rinow theorem.

2. Let $\mathcal{M}$ and $\mathcal{N}$ be two smooth manifolds and $F : \mathcal{M} \to \mathcal{N}$ a smooth map.

   (a) When is F called an *immersion*?

   (b) Let g be a Riemannian metric on $\mathcal{N}$. How is the pullback $(0, 2)$ tensor $F_* g$ defined on $\mathcal{M}$? Show that if F is an immersion, then $F_* g$ is a Riemannian metric on $\mathcal{M}$.

   (c) Let h and g be Riemannian metrics on $\mathcal{M}$ and $\mathcal{N}$, respectively. When is the map F called an *isometry* between $(\mathcal{M}, h)$ and $(\mathcal{N}, g)$?

   (d) Let $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ be the standard 2-torus parametrized by $(\theta^1, \theta^2) \in [0, 2\pi) \times [0, 2\pi)$ and equipped with the flat metric

$$g_{\mathbb{T}^2} = (d\theta^1)^2 + (d\theta^2)^2.$$

   Consider the map $F : \mathbb{T}^2 \to \mathbb{R}^4$ given by

$$F(\theta^1, \theta^2) = \big(\cos(\theta^1), \sin(\theta^1), \cos(\theta^2), \sin(\theta^2)\big).$$

   Show that, if g_E is the standard Euclidean metric on $\mathbb{R}^4$, then $F_* g_E = g_{\mathbb{T}^2}$.

3. Let $\mathcal{M}$ be a smooth manifold.

   (a) What is a connection ∇ on $\mathcal{M}$?

   (b) Show that if ∇ and $\bar{\nabla}$ are two connections on $\mathcal{M}$, then the difference $\nabla - \bar{\nabla}$ is an $(1,2)$-tensor field on $\mathcal{M}$.

   (c) Let g be a Riemannian metric on $\mathcal{M}$. Define the Levi-Civita connection ∇ of $(\mathcal{M}, g)$.

   (d) Let ∇ be the Levi-Civita connection of $(\mathcal{M}, g)$. Show that if X, Y, Z are vector fields on $\mathcal{M}$, then the formula of Koszul is satisfied, i.e.

   $$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle.$$

   (where $\langle \cdot, \cdot \rangle$ denotes $g(\cdot, \cdot)$).

   (e) Show that the Levi-Civita connection ∇ of $(\mathcal{M}, g)$ is unique, i.e. any other connection $\bar{\nabla}$ satisfying the defining properties of ∇ has to coincide with ∇.

   (f) Consider the metric
   $$g = dx^2 + f(x,y)dy^2$$
   for some smooth function $f : \mathbb{R}^2 \to (0, +\infty)$ on $\mathcal{M} = \mathbb{R}^2$. Show that the curves $\{y = \text{const}\}$ are geodesics of $(\mathcal{M}, g)$.

4. Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be Riemannian manifolds and $F : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$ be an isometry. Recall that, in this case, F "commutes" with the covariant derivative, i.e. we have
$$F^*\big(\nabla^{(1)}_X Y\big) = \nabla^{(2)}_{F^*X}(F^*Y) \quad \text{for all} \quad X, Y \in \Gamma(\mathcal{M}_1),$$
where $\nabla^{(i)}$ is the Levi-Civita connection of the metric g_i, $i = 1, 2$.

   (a) Show that if $\gamma_1 : (a, b) \to \mathcal{M}_1$ is a geodesic of $(\mathcal{M}_1, g_1)$, then $\gamma_2 \doteq F \circ \gamma_1$ is a geodesic of $(\mathcal{M}_2, g_2)$.

   (b) We will say that a vector field Z on a Riemannian manifold $(\mathcal{M}, g)$ is *locally constant* if, for every $p \in \mathcal{M}$ and any $X \in T_p\mathcal{M}$, we have
$$\nabla_X Z|_p = 0.$$
   Show that if Z is a locally constant vector field on $\mathcal{M}_1$, then F^*Z is a locally constant vector field on $\mathcal{M}_2$.

   (c) Let $R^{(i)}$ denote the Riemann curvature tensor of $(\mathcal{M}_i, g_i)$, $i = 1, 2$. Show that
$$F^*\big(R^{(1)}(X, Y)Z\big) = R^{(2)}(F^*X, F^*Y)(F^*Z) \quad \text{for all} \;\; X, Y, Z \in \Gamma(\mathcal{M}_1).$$

   Hint: *If needed, you can use without proof the fact that $F^*([X, Y]) = [F^*X, F^*Y]$ for any smooth map F and smooth vector fields X, Y.*

5. Let $\mathcal{M}^m$ be a smooth submanifold of $(\mathcal{N}^n, g)$. Let $\bar{g}$ be the first fundamental form of $\mathcal{M}$. Let us also denote with $\pi^\perp$ and $\pi^\top$ the orthogonal projections onto $(T_p\mathcal{M})^\perp, T_p\mathcal{M} \subset T_p\mathcal{N}$, respectively, for any $p \in \mathcal{M}$.

   (a) Define the *second fundamental form* $B(\cdot,\cdot)$ of $\mathcal{M}$ and show that $B(X,Y) = B(Y,X)$ for all $X, Y \in \Gamma(\mathcal{N},\mathcal{M})$. In the case when $\mathcal{M}$ is of codimension 1, define the *scalar second fundamental form* $b(\cdot,\cdot)$ of $\mathcal{M}$ with respect to a given unit normal $\hat{n}$ to $\mathcal{M}$.

   (b) Recall Gauss's equation relating the Riemann curvature tensor R of the ambient space $(\mathcal{N}, g)$ to that of $(\mathcal{M},\bar{g})$: For all $X, Y, Z, W \in \Gamma(\mathcal{N},\mathcal{M})$,

$$g\big(R(X,Y)Z,W\big) - \bar{g}\big(\bar{R}(X,Y)Z,W\big) = g\big(B(X,Z),B(Y,W)\big) - g\big(B(X,W),B(Y,Z)\big).$$

   Show that if $(\mathcal{N}, g) = (\mathbb{R}^n, g_E)$ and $\mathcal{M}$ is of codimension 1 and admits a unit normal $\hat{n}$, then, for any $p \in \mathcal{M}$ and any 2-plane $\Pi \subset T_p\mathcal{M}$ spanned by two non-collinear tangent vectors X, Y, the sectional curvature $\bar{K}_p(\Pi)$ of $(\mathcal{M},\bar{g})$ satisfies

$$\bar{K}_p(\Pi) = \frac{b(X,X)b(Y,Y) - \big(b(X,Y)\big)^2}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}.$$

   (c) Let $(\mathcal{N}, g) = (\mathbb{R}^3, g_E)$ and $\dim\mathcal{M} = 2$. Show that if $\mathcal{M} \subset \mathbb{R}^n$ contains a straight line passing through p, then

$$\bar{K}_p(T_p\mathcal{M}) \leq 0.$$

   (d) Let $(\mathcal{N}, g) = (\mathbb{R}^{n+1}, g_E)$ and $\dim\mathcal{M} = n \geq 3$. Show that, for every $p \in \mathcal{M}$, there exists a 2-plane $\Pi \subset T_p\mathcal{M}$ such that

$$\bar{K}_p(\Pi) \geq 0.$$

   (as a result, codimension 1 hypersurfaces of $\mathbb{R}^{n+1}$ for $n \geq 3$ cannot have strictly negative sectional curvature).
   Hint: *You can use the symmetry of* $b(\cdot,\cdot)$ *to deduce that the linear map* $L : T_p\mathcal{M} \to \mathcal{M}$ *defined by the relation*

$$\langle Lv, w\rangle = b(v,w) \quad \textit{for all } v, w \in T_p\mathcal{M}$$

   is self-adjoint and hence has a complete basis of orthonormal eigenvectors $\{e_i\}_{i=1}^n$.